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Admissible submonoids of Artin–Tits monoids

Anatole Castella*

Université de Lyon, Université Lyon 1, CNRS UMR 5208 - Institut Camille Jordan, 43 Bd du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France

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Abstract

We show the analogue of Mühlherr's [B. Mühlherr, Coxeter groups in Coxeter groups, in: Finite Geometry and Combinatorics, Cambridge University Press, 1993, pp. 277–287] for Artin–Tits monoids and for Artin–Tits groups of spherical type. That is, the submonoid (resp. subgroup) of an Artin–Tits monoid (resp. group of spherical type) induced by an *admissible partition* of the Coxeter graph is an Artin–Tits monoid (resp. group).

This generalizes and unifies the situation of the submonoid (resp. subgroup) of fixed elements of an Artin–Tits monoid (resp. group of spherical type) under the action of graph automorphisms, and the notion of *LCM-homomorphisms* defined by Crisp in [J. Crisp, Injective maps between Artin groups, in: Geom. Group Theory Down Under (Canberra 1996), de Gruyter, Berlin, 1999, pp. 119–137] and generalized by Godelle in [E. Godelle, Morphismes injectifs entre groupes d'Artin-Tits, Algebr. Geom. Topol. 2 (2002) 519–536].

We then complete the classification of the admissible partitions for which the Coxeter graphs involved have no infinite label, started by Mühlherr in [B. Mühlherr, Some contributions to the theory of buildings based on the gate property, Dissertation, Tübingen, 1994]. This leads us to the classification of Crisp's LCM-homomorphisms.

MSC: 20F36

0. Introduction

In 1993–1994, Mühlherr introduced the notion of *admissible partitions* of a Coxeter graph to define subgroups of the associated Coxeter group that inherit a Coxeter group structure from the ambient one [13,14]. This construction generalizes the situation of the subgroup of fixed elements of a Coxeter group under the action of a group of graph automorphisms, studied by Hée in [11].

The aim of this paper is to show the analogue for Artin–Tits monoids and for Artin–Tits groups of spherical type. Like in the Coxeter case, our construction generalizes the situation of the submonoid (resp. subgroup) of fixed elements of an Artin–Tits monoid (resp. group of spherical type) under the action of a group of graph automorphisms (studied in

* Tel.: +33 (0) 4 72 43 11 49; fax: +33 (0) 4 72 43 16 87.
E-mail address: castella@math.univ-lyon1.fr.

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the early 2000s in [12,7,4,5]). When only finite Coxeter graphs without infinite labels are involved, our construction – more precisely the underlying notion of morphisms between Artin–Tits monoids (or groups) – is equivalent to the notion of *LCM-homomorphisms* defined in 1996 by Crisp [3]. For arbitrary Coxeter graphs, our construction is more general than the notion of LCM-homomorphisms developed in 2002 by Godelle [9], which allowed finite Coxeter graphs with infinite labels, as it works for infinite Coxeter graphs and includes all the morphisms coming from actions of graph automorphisms and all the morphisms induced by the *bursts* of a Coxeter graph used by Paris in [15]. Moreover, we show that some important combinatorial properties of those earlier defined objects (such as their respect of *simple elements* and of *normal forms*) are still valid in our more general context.

We then complete the classification of admissible partitions whose *type* has no infinite label, started by Mühlherr in [14]. With our new point of view on LCM-homomorphisms, this gives us the classification of Crisp's LCM-homomorphisms, started in [3] with the notion of *foldings* of Coxeter graphs (which turn out to be nothing else but special cases of admissible partitions).

1. Preliminaries

1.1. Generalities on monoids

Let M be a monoid, *i.e.* a (non-empty) set endowed with an associative binary operation $M \times M \to M$, $(x, y) \mapsto xy$, with an identity element (denoted by 1). An element $x \in M$ is said to be a *left* (resp. *right*) *unit* if there exists $y \in M$ such that $xy = 1$ (resp. $yx = 1$). For example, 1 is a left and right unit. The monoid M is said to be *left* (resp. *right*) *cancellative* if, for all $x,\ y,\ z \in M$, $xy = xz$ (resp. $yx = zx$) implies $y = z$; and M is said to be *cancellative* if it is left and right cancellative. Note that, in a left or right cancellative monoid, left units and right units coincide.

Let $S = \{s_e \mid e \in E\}$ be a generating subset of M such that the map $E \to S$, $e \mapsto s_e$, is one-to-one. A word $e_1 \cdots e_n$ on E is a *representation* (on E) of $x \in M$ if $x = s_{e_1} \cdots s_{e_n}$, it is called *reduced* if it is of minimal length among all the representations of x. We denote by $\ell_S(x)$ this minimal length, and call the function $\ell_S : M \to \mathbb{N}$ thus defined the *length on M with respect to S*.

We denote by $\preccurlyeq$ (resp. $\succcurlyeq$) the *left* (resp. *right*) *divisibility* in M, *i.e.* for $x,\ y \in M$, we write $y \preccurlyeq x$ (resp. $x \succcurlyeq y$) if there exists $z \in M$ such that $x = yz$ (resp. $x = zy$). There are natural notions of *gcd*'s and *lcm*'s in M: an element d in M is a *left gcd* of a non-empty subset $X \subseteq M$ if $d \preccurlyeq x$ for all $x \in X$ and if, for every $z \in M$ with this property, we get $z \preccurlyeq d$; an element m in M is a *right lcm* of a non-empty subset $X \subseteq M$ if $x \preccurlyeq m$ for all $x \in M$ and if, for every $z \in M$ with this property, we get $m \preccurlyeq z$. The notions of *right gcd* and *left lcm* are defined symmetrically. If two elements $x,\ y \in M$ have a *unique* left (resp. right) lcm, we denote it by $x \vee_L y$ (resp. $x \vee_R y$); and if they have a *unique* left (resp. right) gcd, we denote it by $x \wedge_L y$ (resp. $x \wedge_R y$). Note that in a cancellative monoid with no non-trivial unit, gcd's and lcm's are unique when they exist.

For $x_1, \ldots, x_n \in M$, we denote by $\prod_{k=1}^{n} x_k$ the product $x_1 x_2 \cdots x_n$ in that order. For $x,\ y \in M$ and $n \in \mathbb{N}$, we denote by $\prod_n(x, y)$ the product $xyxy \cdots$ of n terms alternatively equal to x and y (starting with x). If $M = \mathbb{N}$ endowed with the usual addition, we prefer the notation $\sum_n(x, y)$ for the sum $x + y + x + y + \cdots$ of n terms alternatively equal to x and y (starting with x).

1.2. Generalities on Coxeter groups and Artin–Tits groups

Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a *Coxeter matrix* over an arbitrary (non necessarily finite) set I, *i.e.* with $m_{i,j} = m_{j,i} \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}$ and $m_{i,j} = 1 \Leftrightarrow i = j$. The matrix Γ is usually represented by its *Coxeter graph*, *i.e.* the graph with vertex set I, edge set $\{\{i, j\} \mid m_{i,j} \geqslant 3\}$, and a label $m_{i,j}$ over the edge $\{i, j\}$ if $m_{i,j} \geqslant 4$. We denote by

$$W_\Gamma = \left\langle s_i,\ i \in I \mid s_i^2 = 1,\ \prod_{m_{i,j}}(s_i, s_j) = \prod_{m_{i,j}}(s_j, s_i),\ \text{if } m_{i,j} \neq \infty \right\rangle,$$

$$B_\Gamma = \left\langle \boldsymbol{s}_i,\ i \in I \mid \prod_{m_{i,j}}(\boldsymbol{s}_i, \boldsymbol{s}_j) = \prod_{m_{i,j}}(\boldsymbol{s}_j, \boldsymbol{s}_i),\ \text{if } m_{i,j} \neq \infty \right\rangle,$$

$$B_\Gamma^+ = \left\langle \boldsymbol{s}_i,\ i \in I \mid \prod_{m_{i,j}}(\boldsymbol{s}_i, \boldsymbol{s}_j) = \prod_{m_{i,j}}(\boldsymbol{s}_j, \boldsymbol{s}_i),\ \text{if } m_{i,j} \neq \infty \right\rangle^+,$$

the *Coxeter group*, the *Artin–Tits group* and the *Artin–Tits monoid* associated with Γ respectively. Note that we may use the same symbols for the generators of B_Γ and B_Γ^+ since Paris showed in [15] that B_Γ^+ identifies with the submonoid of B_Γ generated by the $\boldsymbol{s}_i$, $i \in I$ (he actually proved this result when I is finite, but this implies the general case). Set $S_\Gamma = \{s_i \mid i \in I\}$ and $\boldsymbol{S}_\Gamma = \{\boldsymbol{s}_i \mid i \in I\}$; we say that the pair (W_Γ, S_Γ) (resp. $(B_\Gamma, \boldsymbol{S}_\Gamma)$, resp. $(B_\Gamma^+, \boldsymbol{S}_\Gamma)$) is the *Coxeter* (resp. *Artin–Tits*, resp. *positive Artin–Tits*) *system* of *type* Γ. Note that W_Γ is generated by S_Γ as a monoid. We denote by the same letter ℓ the lengths on W_Γ with respect to S_Γ, and on B_Γ^+ with respect to $\boldsymbol{S}_\Gamma$, and call them *standard* lengths.

Let $\Gamma = (m_{i,j})_{i,j\in I}$ and $\Gamma' = (m'_{i',j'})_{i',j'\in I'}$ be two Coxeter matrices. An isomorphism from Γ onto Γ' is a bijective map $f : I \to I'$ such that $m_{i,j} = m'_{f(i),f(j)}$ for all $i, j \in I$. In particular, we denote by $\mathrm{Aut}(\Gamma)$ the automorphism group of Γ. We say that two pairs (G_1, S_1) and (G_2, S_2), where G_i is a group (resp. a monoid) generated by S_i ($i = 1, 2$), are *isomorphic* if there exists an isomorphism $f : G_1 \to G_2$ that maps S_1 onto S_2. For example, the two systems (W_Γ, S_Γ) and $(W_{\Gamma'}, S_{\Gamma'})$ (resp. $(B_\Gamma, \boldsymbol{S}_\Gamma)$ and $(B_{\Gamma'}, \boldsymbol{S}_{\Gamma'})$, resp. $(B_\Gamma^+, \boldsymbol{S}_\Gamma)$ and $(B_{\Gamma'}^+, \boldsymbol{S}_{\Gamma'})$) are isomorphic if and only if so are Γ and Γ'.

1.2.1. Simple elements

Let $\pi_\Gamma : B_\Gamma \to W_\Gamma$ be the canonical morphism sending $\boldsymbol{s}_i$ on s_i for all $i \in I$.

The order of $s_i s_j$ in W_Γ is exactly $m_{i,j}$ [1, Ch. V, no 4.3, Prop. 4]. In particular, the map $I \to S_\Gamma$, $i \mapsto s_i$, and hence the map $I \to \boldsymbol{S}_\Gamma$, $i \mapsto \boldsymbol{s}_i$, are one-to-one. Tits showed in [16, Thm. 3] that two reduced representations on I of an element $w \in W$ only differ from a finite sequence of transformations – called *braid relations* – of the form $\prod_{m_{i,j}}(i, j) \rightsquigarrow \prod_{m_{i,j}}(j, i)$ with $i, j \in I$ such that $i \neq j$ and $m_{i,j} \neq \infty$. This property makes the following definition allowable:

Definition 1 (*Simple Elements*). The canonical morphism $\pi_\Gamma : B_\Gamma \to W_\Gamma$ has a section $w \mapsto \boldsymbol{w} \in B_\Gamma^+$ where $\boldsymbol{w}$ is represented on I by one (and hence any) reduced representation of w on I. We say that such an element $\boldsymbol{w}$ in B_Γ^+ is *simple* and set $\boldsymbol{W}_\Gamma = \{\boldsymbol{w} \mid w \in W_\Gamma\} = \{x \in B_\Gamma^+ \mid \ell(x) = \ell(\pi_\Gamma(x))\}$.

1.2.2. Standard parabolicity, sphericity and irreducibility

Let $J \subseteq I$. We set $\Gamma_J = (m_{i,j})_{i,j\in J}$ (it is a Coxeter matrix); and we denote by W_J (resp. B_J, resp. B_J^+) the subgroup of W_Γ (resp. the subgroup of B_Γ, resp. the submonoid of B_Γ^+) generated by $\{s_j \mid j \in J\}$ (resp. $\{\boldsymbol{s}_j \mid j \in J\}$).

Definition 2 (*Standard Parabolicity*). The subgroups W_J (resp. subgroups B_J, resp. submonoids B_J^+), $J \subseteq I$, of W_Γ (resp. B_Γ, resp. B_Γ^+) are called *standard parabolic* (*with respect to* Γ).

The pair $(W_J, \{s_j \mid j \in J\})$ (resp. $(B_J, \{\boldsymbol{s}_j \mid j \in J\})$, resp. $(B_J^+, \{\boldsymbol{s}_j \mid j \in J\})$) is (isomorphic to) the Coxeter (resp. Artin–Tits, resp. positive Artin–Tits) system of type Γ_J (see [1, Ch. IV, no 1.8, Thm. 2] for the Coxeter case, [17, Ch. II, Thm. 4.13] for the Artin–Tits case with I finite – which implies the general result –, the positive Artin–Tits case being obvious).

Moreover, the standard length on W_J (resp. B_J^+) is induced by the one on W_Γ (resp. B_Γ^+) [1, Ch. IV, no 1.8, Cor. 4]. This implies that $\boldsymbol{W}_J = \boldsymbol{W}_\Gamma \cap B_J^+$.

Definition 3 (*Sphericity*). The Coxeter matrix Γ_J is called *spherical* – and the subset J of I is called *spherical* (*with respect to* Γ) – if W_J is finite. In that case, the subgroups W_J, B_J, and submonoid B_J^+ are also called *spherical*.

In a finite Coxeter group, there exists a unique element of maximal standard length, which is of order two if not trivial [1, Ch. IV, Sec. 1, Ex. 22]. If J is spherical, we denote by r_J the unique element of maximal standard length in W_J and by $\boldsymbol{r}_J$ its image in $\boldsymbol{W}_J$ (*i.e.* the unique element of maximal standard length in $\boldsymbol{W}_J$).

Definition 4 (*Irreducibility*). The matrix Γ is said to be *reducible* if there exists a partition of cardinality two $\{J, K\}$ of I such that $m_{j,k} = 2$ for every pair $(j, k) \in J \times K$. In that case, we write $\Gamma = \Gamma_J \times \Gamma_K$, as we have $W_\Gamma = W_J \times W_K$,

$B_\Gamma = B_J \times B_K$ and $B_\Gamma^+ = B_J^+ \times B_K^+$. If this is not the case, then Γ is said to be *irreducible*; this is precisely when the Coxeter graph of Γ is connected.

We assume that the reader is familiar with the list of the irreducible spherical Coxeter graphs, which can be found for example in [1, Ch. VI, no 4.1, Thm. 1].

1.2.3. Properties of B_Γ^+

Since the defining relations of B_Γ^+ are homogeneous, the standard length of B_Γ^+ is *additive*, *i.e.* $\ell(xy) = \ell(x)+\ell(y)$ for all $x,\ y \in B_\Gamma^+$. This clearly implies that B_Γ^+ has no non-trivial unit. Moreover, B_Γ^+ is cancellative [2, Prop. 2.3] (hence gcd's and lcm's are unique when they exist), and two elements of B_Γ^+ always have left and right gcd's, and have a right (resp. left) lcm as soon as they have a right (resp. left) common multiple [2, Props. 4.1 and 4.2].

Example 5. Let $J \subseteq I$ be non-empty. By [2, Thm. 5.6], the elements $\boldsymbol{s}_j$, $j \in J$, have a (left or right) lcm if and only if Γ_J is spherical, and in that case their (left and right) lcm is $\boldsymbol{r}_J$ [2, Prop. 5.7]. In particular, two elements $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$ have a (left or right) lcm if and only if $m_{i,j} \neq \infty$, in which case $\boldsymbol{s}_i \vee_R \boldsymbol{s}_j = \boldsymbol{s}_i \vee_L \boldsymbol{s}_j = \boldsymbol{r}_{\{i,j\}} = \prod_{m_{i,j}}(\boldsymbol{s}_i, \boldsymbol{s}_j) = \prod_{m_{i,j}}(\boldsymbol{s}_j, \boldsymbol{s}_i)$.

In [12, Prop. 2.1], Michel showed that for all $x \in B_\Gamma^+$, there exists a unique maximal (for $\preccurlyeq$) element $L(x)$ in the set $\{\boldsymbol{w} \in \boldsymbol{W}_\Gamma \mid \boldsymbol{w} \preccurlyeq x\}$ of all simple left divisors of x. The maximal simple right divisor $R(x)$ of x is defined symmetrically.

Definition 6 (*Normal Forms*). The *left normal form* of a non-trivial element $x \in B_\Gamma^+$ is the unique sequence $(x_1, \ldots, x_n)$ of elements of $\boldsymbol{W}_\Gamma$ such that $x = x_1 \cdots x_n$, $x_n \neq 1$ and $x_k = L(x_k x_{k+1} \cdots x_n)$ for $1 \leqslant k \leqslant n-1$. *Right normal forms* are defined symmetrically.

It is clear that B_Γ^+ generates B_Γ (as a group). If Γ is spherical, B_Γ is more precisely the *group of fractions* of B_Γ^+, *i.e.* every $b \in B_\Gamma$ can be written $b = x^{-1}y = x'y'^{-1}$ for $x,\ y,\ x',\ y' \in B_\Gamma^+$ [2, Prop. 5.5].

Definition 7 (*Irreducible Fractions*). Assume that Γ is spherical and fix $b \in B_\Gamma$. Then [7, Cor. 7.5] shows that there exists a *unique* pair (x, y) (resp. (x', y')) in $(B_\Gamma^+)^2$ such that $b = x^{-1}y$ and $x \wedge_L y = 1$ (resp. $b = x'y'^{-1}$ and $x' \wedge_R y' = 1$). We say that this pair (x, y) (resp. (x', y')) is an *irreducible left* (resp. *right*) *fraction*, and is the *irreducible left* (resp. *right*) *form* of b.

2. Admissible partitions — The work of Mühlherr

In this section, we recall the definition of an admissible partition of a Coxeter graph and the principal results of [13] on the subgroup of the associated Coxeter group defined by such a partition. Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix and let $W = W_\Gamma$.

2.1. Definitions

Definition 8 (*[13]*). We say that a partition $\tilde{I}$ of I is *spherical* (*with respect to* Γ) – or by abuse of language is a *spherical partition of* Γ – if, for all $\alpha \in \tilde{I}$, Γ_α is spherical (*i.e.* W_α is finite). In that case, we denote by

- $\tilde{S} = \{r_\alpha \mid \alpha \in \tilde{I}\}$ the set of all r_α, $\alpha \in \tilde{I}$ (recall that r_α is the unique element of maximal standard length in W_α),
- $\tilde{W} = \langle \tilde{S} \rangle$ the subgroup of W generated by $\tilde{S}$,
- $\tilde{l} = l_{\tilde{S}}$ the length on $\tilde{W}$ with respect to $\tilde{S}$ ($\tilde{W}$ is generated by $\tilde{S}$ as a monoid),
- $\tilde{\Gamma} = (|r_\alpha r_\beta|)_{\alpha,\beta\in\tilde{I}}$ the Coxeter matrix of orders of the products $r_\alpha r_\beta$ in W. We call $\tilde{\Gamma}$ the *type* of $\tilde{I}$.

Moreover for $\alpha_1, \ldots, \alpha_n \in \tilde{I}$ and $w = \prod_{k=1}^n r_{\alpha_k} \in \tilde{W}$, we say that the word $\prod_{k=1}^n \alpha_k$ on $\tilde{I}$ is *compatible* – or is a *compatible representation of* w – (*with respect to* Γ), if $\ell(w) = \sum_{k=1}^n \ell(r_{\alpha_k})$.

Note that we always have $\ell(w) \leqslant \sum_{k=1}^n \ell(r_{\alpha_k})$, and the equality holds precisely when the representation $R_{\alpha_1} \cdots R_{\alpha_n}$ of w on I, where for $1 \leqslant k \leqslant n$ the word R_{α_k} is a reduced representation of r_{α_k} on I, is reduced.

Notation 9. Let $w \in W$. We set $\begin{cases} I^+(w) = \{i \in I \mid \ell(ws_i) = \ell(w) + 1\} \\ I^-(w) = \{i \in I \mid \ell(ws_i) = \ell(w) - 1\} \end{cases}$.

Note that $I^-(w)$ is a spherical subset of I [13, Lem. 2.8].

Definition 10 (*[13]*). Let $\tilde{I}$ be a partition of I. We say that $\tilde{I}$ is *admissible* (*with respect to* Γ) – or by abuse of language is an *admissible partition of* Γ – if it is a spherical partition of Γ such that, for all $(w, \alpha) \in \tilde{W} \times \tilde{I}$, either $\alpha \subseteq I^+(w)$ or $\alpha \subseteq I^-(w)$.

Remark 11. Let α be a spherical subset of I and $w \in W$. Then $\alpha \subseteq I^-(w)$ (resp. $\alpha \subseteq I^+(w)$) if and only if $\ell(wr_\alpha) = \ell(w) - \ell(r_\alpha)$ (resp. $\ell(wr_\alpha) = \ell(w) + \ell(r_\alpha)$) [13, Lems. 2.4 and 2.8].

2.2. Admissible partitions and Coxeter groups

The two main results of [13] are the following theorems:

Theorem 12 (*[13, Thm. 1.1]*). *Let $\tilde{I}$ be an admissible partition of Γ, of type $\tilde{\Gamma}$. Then the pair $(\tilde{W}, \tilde{S})$ is (isomorphic to) the Coxeter system of type $\tilde{\Gamma}$.*

Theorem 13 (*[13, Thm. 1.2]*). *Let $\tilde{I}$ be a partition of Γ. The following conditions are equivalent:*

(1) *$\tilde{I}$ is an admissible partition of Γ,*
(2) *for all $\alpha, \beta \in \tilde{I}$ with $\alpha \neq \beta$, $\{\alpha, \beta\}$ is an admissible partition of $\Gamma_{\alpha\cup\beta}$.*

So proving the admissibility of a partition reduces to proving the admissibility of partitions of cardinality two. The following lemma gives a criterion for that. It is left as an exercise in [13], but for convenience and because it will be of great importance for our purpose, we prove it below, following [8]. Note that our condition (1b) is slightly weaker than the one of [13, Lem. 3.3]; this formulation simplifies the proof of the second part of the lemma and will be useful later on in Section 3. From now on, we call *2-partition* a partition of cardinality two.

Lemma 14 (*[13, Lem. 3.3]*). *Let $\tilde{I} = \{\alpha, \beta\}$ be a spherical 2-partition of Γ.*

(1) *The following conditions are equivalent:*
    (a) *$\tilde{I}$ is an admissible partition of Γ,*
    (b) *for every integer $0 \leqslant n < |r_\alpha r_\beta| + 1$, the words $\prod_n(\alpha, \beta)$ and $\prod_n(\beta, \alpha)$ are compatible.*
(2) *If Γ is spherical, then $|r_\alpha r_\beta| \neq \infty$ and conditions* (1a) *and* (1b) *above are equivalent to the following condition:*
    (a) *the words $\prod_{|r_\alpha r_\beta|}(\alpha, \beta)$ and $\prod_{|r_\alpha r_\beta|}(\beta, \alpha)$ are compatible.*
    Moreover, we get in that case $\prod_{|r_\alpha r_\beta|}(r_\alpha, r_\beta) = \prod_{|r_\alpha r_\beta|}(r_\beta, r_\alpha) = r_I$.

Proof. The subgroup $\tilde{W} = \langle r_\alpha, r_\beta \rangle$ of W is a dihedral group of order $2|r_\alpha r_\beta|$, hence the reduced representations on $\tilde{I}$ of the elements of $\tilde{W}$ are the words $\prod_n(\alpha, \beta)$ and $\prod_n(\beta, \alpha)$ for every integer $0 \leqslant n < |r_\alpha r_\beta| + 1$.

Suppose (1b) and let us show (1a). Let $w = \prod_n(r_\alpha, r_\beta) \in \tilde{W}$ for some $0 \leqslant n < |r_\alpha r_\beta| + 1$. We have to show that either $\alpha \subseteq I^+(w)$ or $\alpha \subseteq I^-(w)$, and the same for β. We can assume that $w \neq 1$ (because $\alpha \cup \beta = I = I^+(1)$). For $k \in \mathbb{N}$, set $\alpha_k = \alpha$ if k is odd and $\alpha_k = \beta$ if k is even. Since $\prod_n(\alpha, \beta)$ is compatible, we get $\alpha_n \subseteq I^-(w)$. If $|r_\alpha r_\beta| \neq \infty$ and if $n = |r_\alpha r_\beta|$, we thus get by symmetry $\alpha \cup \beta = I = I^-(w)$. If $n < |r_\alpha r_\beta|$, then the word $\prod_{n+1}(\alpha, \beta)$ is compatible, whence $\alpha_{n+1} \subseteq I^-(wr_{\alpha_{n+1}})$ and hence $\alpha_{n+1} \subseteq I^+(w)$.

Suppose (1a) and let us show (1b). We first prove, by induction on $\ell(w)$, that every $w \in \tilde{W}$ admits a compatible representation on $\tilde{I}$. If $w = 1$ this is obvious, else let $i \in I$ be such that $\ell(ws_i) = \ell(w) - 1$. There is no loss of generality in assuming that $i \in \alpha$. Since $\tilde{I}$ is admissible, we have $\alpha \subseteq I^-(w)$, and $\ell(wr_\alpha) = \ell(w) - \ell(r_\alpha)$. By induction, wr_α admits a compatible representation $\alpha_1 \cdots \alpha_n$, and $\alpha_1 \cdots \alpha_n \alpha$ is then a compatible representation of w.

Now, fix an integer $0 \leqslant n < |r_\alpha r_\beta| + 1$ and consider the word $\prod_n(\alpha, \beta)$. If $n < |r_\alpha r_\beta|$, then this word is the unique reduced representation on $\tilde{I}$ of the element $w = \prod_n(r_\alpha, r_\beta) \in \tilde{W}$, so it must be the existing compatible representation of w (it is clear that a non-reduced word on $\tilde{I}$ cannot be compatible). It remains to prove that, if $|r_\alpha r_\beta| \neq \infty$ and if $\prod_{|r_\alpha r_\beta|}(\alpha, \beta)$ is compatible, then so is $\prod_{|r_\alpha r_\beta|}(\beta, \alpha)$. This is clear if $|r_\alpha r_\beta|$ is even, so assume that $|r_\alpha r_\beta|$ is odd and set $w = \prod_{|r_\alpha r_\beta|}(r_\alpha, r_\beta) = \prod_{|r_\alpha r_\beta|}(r_\beta, r_\alpha)$ and $w' = \prod_{|r_\alpha r_\beta|-1}(r_\beta, r_\alpha)$. The word $\prod_{|r_\alpha r_\beta|-1}(\beta, \alpha)$ is the unique reduced representation of w', hence it is compatible and we have $\alpha \subseteq I^-(w')$. Since w' is not the element of maximal standard length in W, we get $\beta \nsubseteq I^-(w')$, whence $\beta \subseteq I^+(w')$ by admissibility, and hence $\prod_{|r_\alpha r_\beta|}(\beta, \alpha)$ is a compatible representation of w.

If Γ is spherical, then it is clear that $|r_\alpha r_\beta| \neq \infty$ and (1b) implies (2a). Conversely, if (2a) holds, then for all $0 \leqslant n < |r_\alpha r_\beta|$, the prefix $\prod_n(\alpha, \beta)$ of $\prod_{|r_\alpha r_\beta|}(\alpha, \beta)$ (resp. $\prod_n(\beta, \alpha)$ of $\prod_{|r_\alpha r_\beta|}(\beta, \alpha)$) is necessarily compatible, whence (1b). Now consider $w = \prod_{|r_\alpha r_\beta|}(r_\alpha, r_\beta) = \prod_{|r_\alpha r_\beta|}(r_\beta, r_\alpha)$ in $\tilde{W}$. Since both words $\prod_{|r_\alpha r_\beta|}(\alpha, \beta)$ and $\prod_{|r_\alpha r_\beta|}(\beta, \alpha)$ are compatible, we get $\alpha \cup \beta = I = I^-(w)$, whence $w = r_I$. □

Let us conclude this subsection with some further properties of admissible partitions:

Propostion 15 (*[13, Prop. 3.5, A1] and [14, Lem. 2.5.5]*). *Let $\tilde{I}$ be an admissible partition of Γ, of type $\tilde{\Gamma}$, and let $w \in \tilde{W}$.*

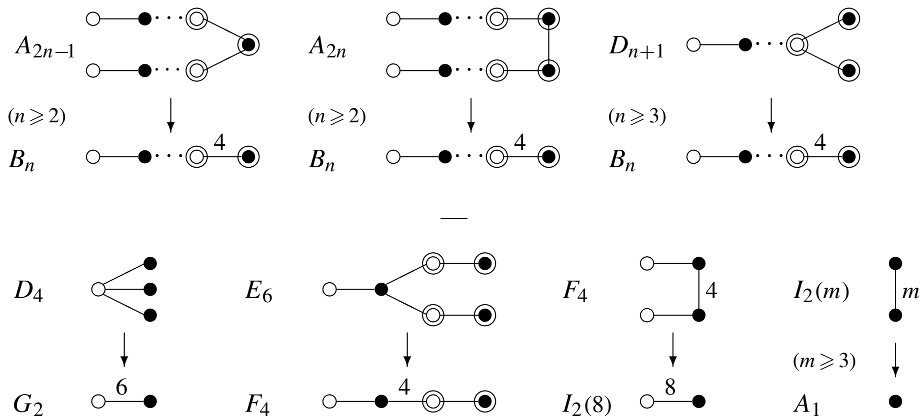
(1) *a representation of w on $\tilde{I}$ is reduced if and only if it is compatible,*
(2) *Γ is spherical if and only if so is $\tilde{\Gamma}$, in which case $r_I = r_{\tilde{I}}$.*

2.3. Examples

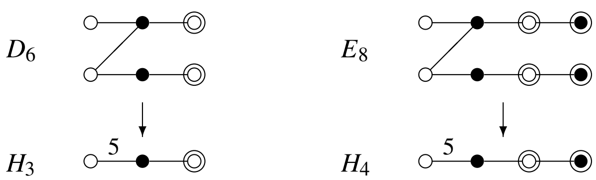
Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix and G be a subgroup of $\mathrm{Aut}(\Gamma)$. The action of G on I induces an action of G on W_Γ which preserves the standard length. If α is an orbit of I under G, then G stabilizes W_α and hence, if α is spherical, G fixes r_α (which is the unique element of maximal standard length in W_α). So if we denote by $\tilde{J}$ the set of spherical orbits of I under G, by $J = \bigcup_{\alpha\in\tilde{J}} \alpha \subseteq I$ their union and if we set $\tilde{S} = \{r_\alpha \mid \alpha \in \tilde{J}\}$ and $\tilde{W} = \langle\tilde{S}\rangle$, we get that $\tilde{W}$ is included in the subgroup $(W_\Gamma)^G$ of fixed points of W_Γ under G, and that $\tilde{J}$ is an admissible partition of Γ_J. Let $\tilde{\Gamma}$ be the type of $\tilde{J}$.

In fact, it can be shown that $\tilde{W} = (W_\Gamma)^G$, hence $((W_\Gamma)^G, \tilde{S})$ is (isomorphic to) the Coxeter system of type $\tilde{\Gamma}$ [13, Thm. 1.3]. See [11, Cor. 3.5] for the original proof of that result.

Example 16. The non-trivial automorphisms of the spherical irreducible Coxeter graphs, and the type of the different sets of orbits we get are symbolized here (see [14, Sec. 2.5] or Section 4 below for justifications):



Example 17. Two admissible partitions that are not the set of orbits of an action of graph automorphisms are given here (see [14, Sec. 2.5], Section 3.3.3 or Section 4 below for justifications):



3. Admissible partitions and Artin–Tits monoids or groups

In Section 3.2 below, we introduce the submonoid of an Artin–Tits monoid (resp. the subgroup of an Artin–Tits group), and the morphism between Artin–Tits monoids or groups, induced by an admissible partition of a Coxeter graph, and we establish the analogue of [13, Thm. 1.1] (*cf.* Theorem 12 above) for Artin–Tits monoids and for Artin–Tits groups of spherical type.

In Section 3.3, we explain how our constructions generalize the situations of the submonoids (resp. subgroups) of fixed elements of an Artin–Tits monoid (resp. group of spherical type) under the action of graph automorphisms, of the LCM-homomorphisms [3,9], and of the morphisms between Artin–Tits monoids (or groups) induced by the *bursts* of a Coxeter graph [15].

In Section 3.4, we show that some important properties of submonoids of fixed elements of an Artin–Tits monoid under the action of graph automorphisms and of LCM-homomorphisms extend to our settings. In particular, we establish them for the morphisms induced by the *bursts* of a Coxeter graph [15], for which they were not known when Coxeter graphs with infinite labels are involved.

But let us begin this section by recalling the notion of *morphisms that respect lcm's* defined by Crisp in [3]. It is the key tool in the proofs of the injectivity of the LCM-homomorphisms in [3,9], and plays a similar role for our main result of Section 3.2.

3.1. Morphisms that respect lcm's

Let $\Gamma = (m_{i,j})_{i,j\in I}$ and $\tilde{\Gamma} = (\tilde{m}_{\alpha,\beta})_{\alpha,\beta\in\tilde{I}}$ be two Coxeter matrices (where $\tilde{I}$ is here an arbitrary set). If x and y are two elements of B^+_Γ (resp. $B^+_{\tilde{\Gamma}}$), we say for short that $x \vee_R y$ *exists* in B^+_Γ (resp. $B^+_{\tilde{\Gamma}}$) to state that x and y admit a right lcm in B^+_Γ (resp. $B^+_{\tilde{\Gamma}}$).

Definition 18 (*[3, Def. 1.1]*). We say that a morphism $\varphi : B^+_{\tilde{\Gamma}} \to B^+_\Gamma$ *respects right lcm's* if:

(1) for all $\alpha \in \tilde{I}$, $\varphi(\boldsymbol{s}_\alpha) \neq 1$,
(2) for all $\alpha, \beta \in \tilde{I}$, $\boldsymbol{s}_\alpha \vee_R \boldsymbol{s}_\beta$ exists in $B^+_{\tilde{\Gamma}}$ if and only if $\varphi(\boldsymbol{s}_\alpha) \vee_R \varphi(\boldsymbol{s}_\beta)$ exists in B^+_Γ, in which case $\varphi(\boldsymbol{s}_\alpha) \vee_R \varphi(\boldsymbol{s}_\beta) = \varphi(\boldsymbol{s}_\alpha \vee_R \boldsymbol{s}_\beta)$.

Morphisms that *respect left lcm's* are defined symmetrically, and we say that such a morphism *respects lcm's* if it respects right and left lcm's.

Propostion 19 (*[4, Thm. 8]*). *Let $\varphi : B^+_{\tilde{\Gamma}} \to B^+_\Gamma$ be a morphism that respects right lcm's. Then:*

(1) *for all $x, y \in B^+_{\tilde{\Gamma}}$, $x \vee_R y$ exists in $B^+_{\tilde{\Gamma}}$ if and only if $\varphi(x) \vee_R \varphi(y)$ exists in B^+_Γ, in which case $\varphi(x) \vee_R \varphi(y) = \varphi(x \vee_R y)$,*
(2) *for all $x, y \in B^+$, $\varphi(x) \preccurlyeq \varphi(y) \Rightarrow x \preccurlyeq y$. In particular, φ is injective.*

Of course, the symmetrical version of Proposition 19 is also true. Here is a fundamental example of morphism that respects lcm's (*cf.* [3,9] and Theorem 23 below):

Lemma 20. *Let $(J_\alpha)_{\alpha\in\tilde{I}}$ be a family of non-empty spherical subsets of I and assume that, for all $\alpha, \beta \in \tilde{I}$, $\tilde{m}_{\alpha,\beta} \neq \infty$ implies that $\Gamma_{J_\alpha\cup J_\beta}$ is spherical and $\boldsymbol{r}_{J_\alpha\cup J_\beta} = \prod_{\tilde{m}_{\alpha,\beta}}(\boldsymbol{r}_{J_\alpha}, \boldsymbol{r}_{J_\beta})$. Then the map $\boldsymbol{s}_\alpha \mapsto \boldsymbol{r}_{J_\alpha}$ extends to a morphism from $B^+_{\tilde{\Gamma}}$ to B^+_Γ. Moreover, if for all $\alpha, \beta \in \tilde{I}$, $\tilde{m}_{\alpha,\beta} = \infty$ implies that $\Gamma_{J_\alpha\cup J_\beta}$ is non-spherical, then this morphism respects lcm's.*

Proof. The first point is clear since the hypothesis implies $\prod_{\tilde{m}_{\alpha,\beta}}(\boldsymbol{r}_{J_\alpha}, \boldsymbol{r}_{J_\beta}) = \prod_{\tilde{m}_{\alpha,\beta}}(\boldsymbol{r}_{J_\beta}, \boldsymbol{r}_{J_\alpha})$ if $\tilde{m}_{\alpha,\beta} \neq \infty$. Let us show the second point. We get $\varphi(\boldsymbol{s}_\alpha) = \boldsymbol{r}_{J_\alpha} \neq 1$ since J_α is non-empty. Moreover, we have the following sequence of equivalences (where the symbol $\vee$ stands for $\vee_L$ or $\vee_R$): $\boldsymbol{s}_\alpha \vee \boldsymbol{s}_\beta$ exists in $B^+_{\tilde{\Gamma}} \Leftrightarrow \tilde{m}_{\alpha,\beta} \neq \infty \Leftrightarrow \Gamma_{J_\alpha\cup J_\beta}$ is spherical $\Leftrightarrow \boldsymbol{r}_{J_\alpha\cup J_\beta} = \boldsymbol{r}_{J_\alpha} \vee \boldsymbol{r}_{J_\beta}$ exists in B^+_Γ, in which case we get $\varphi(\boldsymbol{s}_\alpha \vee \boldsymbol{s}_\beta) = \varphi(\prod_{\tilde{m}_{\alpha,\beta}}(\boldsymbol{s}_\alpha, \boldsymbol{s}_\beta)) = \prod_{\tilde{m}_{\alpha,\beta}}(\boldsymbol{r}_{J_\alpha}, \boldsymbol{r}_{J_\beta}) = \boldsymbol{r}_{J_\alpha\cup J_\beta} = \boldsymbol{r}_{J_\alpha} \vee \boldsymbol{r}_{J_\beta} = \varphi(\boldsymbol{s}_\alpha) \vee \varphi(\boldsymbol{s}_\beta)$. $\square$

3.2. Admissible morphisms, submonoids and subgroups

Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix.

The admissibility of a spherical partition $\tilde{I}$ of Γ can naturally be expressed in terms of simple elements in B_Γ^+. Indeed, if we denote by $\tilde{\boldsymbol{W}}$ the image of the subgroup $\tilde{W} = \langle r_\alpha \mid \alpha \in \tilde{I}\rangle$ of W_Γ in $\boldsymbol{W}_\Gamma \subseteq B_\Gamma^+$, then we get that $\tilde{I}$ is admissible if and only if, for all $(\boldsymbol{w}, \alpha) \in \tilde{\boldsymbol{W}} \times \tilde{I}$, either the products $\boldsymbol{w} \cdot \boldsymbol{s}_i$ are simple for all $i \in \alpha$, or $\boldsymbol{w} \succcurlyeq \boldsymbol{s}_i$ for all $i \in \alpha$. In the same way, the compatibility of words on $\tilde{I}$ is easy to characterize:

Lemma 21. *let $\tilde{I}$ be a spherical partition of Γ and fix $\alpha_1, \dots, \alpha_n \in \tilde{I}$. Then*

$$\text{the word } \prod_{k=1}^{n} \alpha_k \text{ is compatible } \Leftrightarrow \text{ the element } \prod_{k=1}^{n} \boldsymbol{r}_{\alpha_k} \text{ is simple.}$$

In that case, if $w = \prod_{k=1}^n r_{\alpha_k}$ in $\tilde{W}$, then $\boldsymbol{w} = \prod_{k=1}^n \boldsymbol{r}_{\alpha_k}$ in $\boldsymbol{W}_\Gamma$.

Proof. Set $w = \prod_{k=1}^n r_{\alpha_k} = \pi\left(\prod_{k=1}^n \boldsymbol{r}_{\alpha_k}\right)$. Assume that $\prod_{k=1}^n \alpha_k$ is compatible, *i.e.* $\ell(w) = \sum_{k=1}^n \ell(r_{\alpha_k})$, and fix a reduced representation R_{α_k} of each r_{α_k} on I. Then the representation $\prod_{k=1}^n R_{\alpha_k}$ of w on I is reduced and hence, by definition of $\boldsymbol{w}$, we get $\boldsymbol{w} = \prod_{k=1}^n \boldsymbol{r}_{\alpha_k}$ in $\boldsymbol{W}_\Gamma$. Conversely, if the product $\prod_{k=1}^n \boldsymbol{r}_{\alpha_k}$ is simple, then $\ell(w) = \ell\left(\prod_{k=1}^n \boldsymbol{r}_{\alpha_k}\right) = \sum_{k=1}^n \ell(\boldsymbol{r}_{\alpha_k}) = \sum_{k=1}^n \ell(r_{\alpha_k})$ (the first and third equalities by definition of $\boldsymbol{W}_\Gamma$, and the second by additivity of the standard length on B_Γ^+), whence the compatibility of $\prod_{k=1}^n \alpha_k$. □

This lemma allows us to reformulate the characterizations of the admissibility of a 2-partition of Γ (*cf.* Lemma 14 above) in terms of simple elements of B_Γ^+:

Lemma 22. *Let $\tilde{I} = \{\alpha, \beta\}$ be a spherical 2-partition of Γ.*

(1) *The following conditions are equivalent:*
 (a) *$\tilde{I}$ is an admissible partition of Γ,*
 (b) *for every integer $0 \leqslant n < |r_\alpha r_\beta| + 1$, the two elements $\prod_n(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta)$ and $\prod_n(\boldsymbol{r}_\beta, \boldsymbol{r}_\alpha)$ of B_Γ^+ are simple.*

(2) *If Γ is spherical, then $|r_\alpha r_\beta| \neq \infty$ and conditions* (1a) *and* (1b) *above are equivalent to the following:*
 (a) *the elements $\prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta)$ and $\prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_\beta, \boldsymbol{r}_\alpha)$ of B_Γ^+ are simple.*
 Moreover, we get in that case $\prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta) = \prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_\beta, \boldsymbol{r}_\alpha) = \boldsymbol{r}_I$.

We are now able to prove the analogue of Theorem 12 for Artin–Tits monoids and for Artin–Tits groups of spherical type:

Theorem 23. *Let $\tilde{I}$ be an admissible partition of Γ, of type $\tilde{\Gamma}$. Then:*

(1) *the map $\boldsymbol{S}_{\tilde{\Gamma}} \to B_\Gamma^+$, $\boldsymbol{s}_\alpha \mapsto \boldsymbol{r}_\alpha$, extends to a morphism $\varphi = \varphi_{\tilde{I}} : B_{\tilde{\Gamma}}^+ \to B_\Gamma^+$,*
(2) *this morphism respects lcm's, hence is injective.*

In particular, if we set $\tilde{\boldsymbol{S}} = \{\boldsymbol{r}_\alpha \mid \alpha \in \tilde{I}\}$ and denote by $\tilde{B}^+ = \langle \tilde{\boldsymbol{S}}\rangle^+$ the submonoid of B_Γ^+ generated by the $\boldsymbol{r}_\alpha$, $\alpha \in \tilde{I}$, then the pair $(\tilde{B}^+, \tilde{\boldsymbol{S}})$ is (isomorphic to) the positive Artin–Tits system of type $\tilde{\Gamma}$.

Proof. We can apply Lemma 20 to the set $\tilde{I}$, since it consists of non-empty spherical subsets of I, and since we have $|r_\alpha r_\beta| \neq \infty$ if and only if $\Gamma_{\alpha\cup\beta}$ is spherical (by Proposition 15), in which case we get $\prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta) = \boldsymbol{r}_{\alpha\cup\beta}$ by Lemma 22. □

The morphism $\varphi : B_{\tilde{\Gamma}}^+ \hookrightarrow B_\Gamma^+$ of Theorem 23 clearly extends to a group homomorphism $\varphi_{\mathrm{gr}} : B_{\tilde{\Gamma}} \to B_\Gamma$ whose image is the subgroup $\tilde{B} = \langle \boldsymbol{r}_\alpha,\ \alpha \in \tilde{I}\rangle$ of B_Γ. When $\tilde{\Gamma}$ is spherical, the injectivity of φ implies the following:

Theorem 24. *Let $\tilde{I}$ be an admissible partition of Γ, of spherical type $\tilde{\Gamma}$. Then the homomorphism $\varphi_{\mathrm{gr}} : B_{\tilde{\Gamma}} \to B_\Gamma$ is injective. In other words, the pair $(\tilde{B}, \tilde{\boldsymbol{S}})$ is (isomorphic to) the Artin–Tits system of type $\tilde{\Gamma}$.*

Proof. Since $\tilde{\Gamma}$ is spherical, every $b \in B_{\tilde{\Gamma}}$ can be written $b = x^{-1}y$ for $x,\ y \in B_{\tilde{\Gamma}}^+$ (*cf.* Section 1.2), and the equality $\varphi_{\mathrm{gr}}(b) = 1$ hence implies $\varphi(x) = \varphi(y)$, whence the result thanks to the injectivity of φ. □

Let us name the objects we have just defined:

Definition 25. Let $J \subseteq I$ be a subset of I and let $\tilde{J}$ be an admissible partition of Γ_J, of type $\tilde{\Gamma}$. Let $\tilde{\boldsymbol{S}} = \{\boldsymbol{s}_\alpha \mid \alpha \in \tilde{J}\}$. Then we say that:

- the submonoid $\tilde{B}^+ = \langle \tilde{\boldsymbol{S}} \rangle^+$ of B_Γ^+ (resp. the subgroup $\tilde{B} = \langle \tilde{\boldsymbol{S}} \rangle$ of B_Γ) is *induced* by $\tilde{J}$, or, by abuse of language, is an *admissible* submonoid (resp. subgroup) of B_Γ^+ (resp. B_Γ),
- the morphism $\varphi = \varphi_{\tilde{J}} : B_{\tilde{\Gamma}}^+ \hookrightarrow B_\Gamma^+$ (resp. $\varphi_{\mathrm{gr}} : B_{\tilde{\Gamma}} \to B_\Gamma$), which sends each $\boldsymbol{s}_\alpha \in \boldsymbol{S}_{\tilde{\Gamma}}$ on $\boldsymbol{r}_\alpha \in \tilde{\boldsymbol{S}}$, is *induced* by $\tilde{J}$, or, by abuse of language, is an *admissible* morphism.

Remark 26. In our definitions, we allow partitions of *subsets* of I. This generalization does not change the conclusions of Theorems 23 and 24, and allows the notion of admissible submonoids, subgroups or morphisms, to comprise the notions of standard parabolic submonoids or subgroups, of submonoids of fixed elements under the action of graph automorphisms and of LCM-homomorphisms of [3,9] (see Theorems 28 and 33 below).

Remark 27. If the partition $\tilde{J}$ of Γ_J is only supposed to be spherical, then the map $\boldsymbol{S}_{\tilde{\Gamma}} \to B_\Gamma^+$, $\boldsymbol{s}_\alpha \mapsto \boldsymbol{r}_\alpha$, does not necessarily extend to a morphism from $B_{\tilde{\Gamma}}^+$ to B_Γ^+: for example, if $\Gamma = \overset{1}{\circ}\!-\!\!-\!\!-\overset{2}{\bullet}\!-\!\!-\!\!-\overset{3}{\bullet}$ with $\alpha = \{1\}$ and $\beta = \{2, 3\}$, then $|r_\alpha r_\beta| = 3$ but $\boldsymbol{r}_\alpha \boldsymbol{r}_\beta \boldsymbol{r}_\alpha \neq \boldsymbol{r}_\beta \boldsymbol{r}_\alpha \boldsymbol{r}_\beta$ in B_Γ^+ (look at the standard length).

3.3. Admissibility and Artin–Tits monoids or groups in the literature

In this subsection, we show how our notions of admissible submonoids, subgroups or morphisms generalize and unify three situations that have been studied earlier.

3.3.1. Submonoids of fixed points under the action of graph automorphisms

Here is the analogue of [11, Cor. 3.5] and [13, Thm. 1.3] (*cf.* Section 2.3 above) for Artin–Tits monoids and for Artin–Tits groups of spherical type. Hence we recover the results [7, Thm. 9.3], [12, Cor. 4.4] and [4, Lem. 10 and Thm. 11].

Theorem 28. *Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix and G be a subgroup of* $\mathrm{Aut}(\Gamma)$. *Let $\tilde{J}$ be the set of all spherical orbits of I under G and let $J \subseteq I$ be their union. Let $\tilde{\Gamma}$ be the type of the admissible partition $\tilde{J}$ of Γ_J, and set $\tilde{\boldsymbol{S}} = \{\boldsymbol{r}_\alpha \mid \alpha \in \tilde{J}\}$, $\tilde{B}^+ = \langle \tilde{\boldsymbol{S}} \rangle^+$ and $\tilde{B} = \langle \tilde{\boldsymbol{S}} \rangle$. Then:*

(1) *$(B_\Gamma^+)^G = \tilde{B}^+$ and hence the pair $((B_\Gamma^+)^G, \tilde{\boldsymbol{S}})$ is (isomorphic to) the positive Artin–Tits system of type $\tilde{\Gamma}$,*

(2) *if Γ is spherical, then $(B_\Gamma)^G = \tilde{B}$ and hence the pair $((B_\Gamma)^G, \tilde{\boldsymbol{S}})$ is (isomorphic to) the Artin–Tits system of type $\tilde{\Gamma}$.*

Proof. We already know that $\tilde{J}$ is an admissible partition of Γ_J (*cf.* Section 2.3). Thanks to Theorems 23 and 24 above, the only things to prove are $(B_\Gamma^+)^G = \tilde{B}^+$ and, when Γ is spherical, $(B_\Gamma)^G = \tilde{B}$.

For $\alpha \in \tilde{J}$, the group G stabilizes B_α^+ and the induced action respects the standard length, so G fixes $\boldsymbol{r}_\alpha$ (which is the unique element of maximal standard length in $\boldsymbol{W}_\alpha$). Hence we get $\tilde{B}^+ \subseteq (B_\Gamma^+)^G$ and $\tilde{B} \subseteq (B_\Gamma)^G$.

Let x be an element of $(B_\Gamma^+)^G$ and let us show by induction on $\ell(x)$ that $x \in \tilde{B}^+$. There is nothing to prove if $x = 1$, so assume that $x \neq 1$ and consider an element $i \in I$ such that $\boldsymbol{s}_i \preccurlyeq x$. Then, for all $g \in G$, $\boldsymbol{s}_{g(i)} \preccurlyeq x$. This implies that the orbit α of i under G is spherical and that $\boldsymbol{r}_\alpha \preccurlyeq x$. So there exists $x' \in B_\Gamma^+$ such that $x = \boldsymbol{r}_\alpha x'$, and $\ell(x') < \ell(x)$. By cancellativity in B_Γ^+, we get $x' \in (B_\Gamma^+)^G$, hence $x' \in \tilde{B}^+$ by induction, and finally $x \in \tilde{B}^+$.

Now assume that Γ is spherical and fix $b \in (B_\Gamma)^G$. Let $(x, y) \in (B_\Gamma^+)^2$ be the irreducible left form of b (i.e. the unique pair such that $b = x^{-1} y$ and $x \wedge_L y = 1$, *cf.* Definition 7 above). Since the action of G on B_Γ^+ respect divisibility (hence gcd's), we get by unicity that $x, y \in (B_\Gamma^+)^G$. The first point then gives $x, y \in \tilde{B}^+$, whence $b \in \tilde{B}$. $\square$

Remark 29. On the work of Crisp [4].

(1) Our proof of Theorem 28 is very similar to those of [4, Lem. 10 and Thm. 11], and indeed, the results [4, Lem. 6], [5] and Lemma 32 below show that the Coxeter matrix $(m_{BC})_{B,C\in\mathbf{S}}$ constructed by Crisp in [4,5] is precisely our matrix $\tilde{\Gamma}$.
(2) Crisp actually established the second point of Theorem 28 for a wider class of Coxeter graphs than the spherical ones, namely the *type FC* ones, *i.e.* the finite Coxeter graphs for which every complete subgraph with no infinite label is spherical [4, Thm. 4].

3.3.2. LCM-homomorphisms

We recall in Definition 31 below the notion of *LCM-homomorphisms* of [9, Def. 2.1], which generalizes the one of [3, Def. 2.1] by allowing finite Coxeter graphs with infinite labels. We adapt these definitions to our settings by defining the notion of *LCM-partitions* of a Coxeter graph, which will turn out to be nothing else but special cases of admissible partitions (*cf.* Theorem 33 below). We do not suppose that the Coxeter graphs involved are finite.

Definition 30. Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix and let $\tilde{I}$ be a spherical partition of Γ. Let $\Omega = (n_{\alpha,\beta})_{\alpha,\beta\in\tilde{I}}$ be a Coxeter matrix over $\tilde{I}$. We say that $\tilde{I}$ is an *LCM-partition* of Γ, of *type* Ω, if, for each pair $(\alpha,\beta)\in\tilde{I}^2$, we have the following alternative:

(Fi) $n_{\alpha,\beta}\neq\infty$, $\Gamma_{\alpha\cup\beta}$ is spherical and $\boldsymbol{r}_{\alpha\cup\beta} = \prod_{n_{\alpha,\beta}}(\boldsymbol{r}_\alpha,\boldsymbol{r}_\beta)$,
(In) $n_{\alpha,\beta}=\infty$ and for all $i\in\alpha$, $\Gamma_{\{i\}\cup\beta}$ is non-spherical.

Definition 31 (*[3,9, Defs. 2.1]*). Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix. Let $J\subseteq I$ be a subset of I and let $\tilde{J}$ be an LCM-partition of Γ_J, of type $\Omega = (n_{\alpha,\beta})_{\alpha,\beta\in\tilde{J}}$. Lemma 20 above shows that the map $\boldsymbol{S}_\Omega\to B^+_\Gamma$, $\boldsymbol{s}_\alpha\mapsto\boldsymbol{r}_\alpha$, extends to a morphism that respects lcm's from B^+_Ω to B^+_Γ, which we call, after [3,9], an *LCM-homomorphism*.

Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix, and let $\tilde{I}$ be an LCM-partition of Γ, of type $\Omega = (n_{\alpha,\beta})_{\alpha,\beta\in\tilde{I}}$. We show in Theorem 33 below that $\tilde{I}$ is an admissible partition of Γ, and that its type (as an LCM-partition) Ω is necessarily its type (as a spherical partition) $\tilde{\Gamma} = (|r_\alpha r_\beta|)_{\alpha,\beta\in\tilde{I}}$.

Lemma 32. *Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix and let α and β be two spherical subsets of I.*

(1) *If $\Gamma_{\alpha\cup\beta}$ is spherical and if there exists an integer $n\in\mathbb{N}$ such that $\boldsymbol{r}_{\alpha\cup\beta} = \prod_n(\boldsymbol{r}_\alpha,\boldsymbol{r}_\beta) = \prod_n(\boldsymbol{r}_\beta,\boldsymbol{r}_\alpha)$, then $n=|r_\alpha r_\beta|$.*
(2) *If, for all $n\in\mathbb{N}$, the product $\prod_n(\boldsymbol{r}_\alpha,\boldsymbol{r}_\beta)$ is simple, then $|r_\alpha r_\beta|=\infty$ and $\Gamma_{\alpha\cup\beta}$ is non-spherical.*

Proof. Under the hypothesis of assertion (1), we get $(r_\alpha r_\beta)^n = \prod_{2n}(r_\alpha,r_\beta) = (r_{\alpha\cup\beta})^2 = 1$ in W_Γ, hence $|r_\alpha r_\beta|$ divides n. If $|r_\alpha r_\beta| < n$, then we can replace a factor $\prod_{|r_\alpha r_\beta|}(r_\alpha,r_\beta)$ of $\prod_n(r_\alpha,r_\beta)$ by $\prod_{|r_\alpha r_\beta|}(r_\beta,r_\alpha)$ and then simplify $2|r_\alpha r_\beta|$ terms, whence $\ell\left(\prod_n(r_\alpha,r_\beta)\right) < \sum_n(\ell(r_\alpha),\ell(r_\beta)) = \sum_n(\ell(\boldsymbol{r}_\alpha),\ell(\boldsymbol{r}_\beta)) = \ell(\prod_n(\boldsymbol{r}_\alpha,\boldsymbol{r}_\beta))$, and a contradiction since $\prod_n(\boldsymbol{r}_\alpha,\boldsymbol{r}_\beta)$ is simple. Under the hypothesis of assertion (2), the dihedral group $\langle r_\alpha,r_\beta\rangle$, which is included in $W_{\alpha\cup\beta}$, is infinite, hence $|r_\alpha r_\beta|=\infty$ and $\Gamma_{\alpha\cup\beta}$ is non-spherical. $\square$

Theorem 33. *Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix, and let $\tilde{I}$ be an LCM-partition of Γ, of type $\Omega = (n_{\alpha,\beta})_{\alpha,\beta\in\tilde{I}}$. Then $\tilde{I}$ is an admissible partition of Γ, and $\Omega=\tilde{\Gamma}=(|r_\alpha r_\beta|)_{\alpha,\beta\in\tilde{I}}$.*

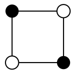
Proof. A consequence of [9, Lem. 2.5] is that, if $n_{\alpha,\beta}=\infty$, then for all $n\in\mathbb{N}$, the product $\prod_n(\boldsymbol{r}_\alpha,\boldsymbol{r}_\beta)$ is simple. Lemma 32 then shows that $\Omega=\tilde{\Gamma}$ and the characterizations of Lemma 22 show that for all $\alpha,\beta\in\tilde{I}$, $\{\alpha,\beta\}$ is an admissible partition of $\Gamma_{\alpha\cup\beta}$. We conclude that $\tilde{I}$ is an admissible partition of Γ thanks to Theorem 13. $\square$

So, as announced, an LCM-partition is an admissible partition (and hence an LCM-homomorphism is an admissible morphism); the converse is false in general (*cf.* Example 34, Remark 39 and Example 45 below), but is true for example if:

(1) the matrix $\tilde{\Gamma}$ has no infinite coefficient,
(2) the matrix Γ is *right angled*, *i.e.* $m_{i,j}\in\{1,2,\infty\}$ for all $i,j\in I$ (to see this, use [14, Lem. 2.5.15], recalled in Proposition 48 below),

(3) the matrix Γ is of type FC (this notion is defined in Remark 29) and $\tilde{I}$ is the set of orbits of I under the action of a subgroup of $\mathrm{Aut}(\Gamma)$.

Example 34. Consider the Coxeter graph Γ of affine type $\tilde{A}_3$, and its 2-partition formed by pairs of opposite vertices:



This spherical 2-partition is admissible since it is the set of orbits of Γ under the action of the "central symmetry", and its type is $\tilde{\Gamma} = I_2(\infty)$ since Γ is non-spherical. It is not an LCM-partition (condition (In) of Definition 30 is not satisfied): indeed, if i is one of the vertices of Γ and if β is the orbit that does not contain i, then $\Gamma_{\{i\}\cup\beta}$ is of spherical type A_3.

Remark 35. The results [3, Prop. 2.3] and [9, Cor. 2.7] on the injectivity of the morphism between Coxeter groups induced by an LCM-homomorphism now appear as special cases of [13, Thm. 1.1] (recalled in Theorem 12 above). In fact, one can check that the proof of [9, Cor. 2.7] works for general admissible partitions and hence gives a new proof of [13, Thm. 1.1].

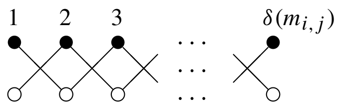
3.3.3. The bursts of a Coxeter graph

We recall here a construction of Mühlherr [14, Sec. 2.6], a quasi-identical version of which has independently been obtained by Crisp and Paris for Coxeter graphs with no infinite label [6, Sec. 6], and by Paris in general [15, Sec. 5]. The differences between the two approaches rely essentially in the choice of the integer N in Definition 36 below.

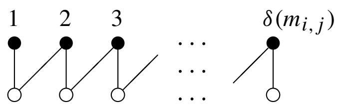
$$\text{Let } \delta : \mathbb{N}_{\geqslant 2} \cup \{\infty\} \to \mathbb{N}_{\geqslant 1}, m \mapsto \begin{cases} m-1 & \text{if } m \text{ is even ,} \\ \dfrac{m-1}{2} & \text{if } m \text{ is odd,} \\ 2 & \text{if } m = \infty. \end{cases}$$

Definition 36 (*[14, Sec. 2.6]*). Suppose that $\Gamma = (m_{i,j})_{i,j\in I}$ is a Coxeter matrix such that the subset $\{m_{i,j} \mid i,\ j \in I\}$ of $\mathbb{N} \cup \{\infty\}$ is finite. Set $N_0 = \mathrm{lcm}\{\delta(m_{i,j}) \mid i,\ j \in I,\ i \neq j\}$ and let N be a multiple of N_0. A *N-burst*, or simply a *burst*, of Γ is a Coxeter graph $\widehat{\Gamma}$ with vertex set the disjoint union $\widehat{I} = \bigsqcup_{i\in I} T(i)$ of sets $T(i) = \{i^{(1)}, \dots, i^{(N)}\}$ of cardinality N, and with edges displayed as follows:

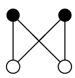
(1) there is no edge between two elements of a same $T(i)$,
(2) if $m_{i,j} \in \mathbb{N}_{\geqslant 2}$ is even, the graph $\widehat{\Gamma}_{T(i)\sqcup T(j)}$ is the disjoint union of $\frac{N}{\delta(m_{i,j})}$ copies of the following graph:



where the vertices $\bullet$ constitute $T(i)$ and the vertices $\circ$ constitute $T(j)$,
(3) if $m_{i,j} \in \mathbb{N}_{\geqslant 3}$ is odd, the graph $\widehat{\Gamma}_{T(i)\sqcup T(j)}$ is the disjoint union of $\frac{N}{\delta(m_{i,j})}$ copies of the following graph:



where the vertices $\bullet$ constitute $T(i)$ and the vertices $\circ$ constitute $T(j)$,
(4) if $m_{i,j} = \infty$, the graph $\widehat{\Gamma}_{T(i)\sqcup T(j)}$ is the disjoint union of $\frac{N}{\delta(m_{i,j})}$ copies of the following graph:



where the vertices $\bullet$ constitute $T(i)$ and the vertices $\circ$ constitute $T(j)$.

Theorem 37 (*[14, Thm. 2.6.1 and its proof]*). *Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix with $\{m_{i,j} \mid i,\ j \in I\}$ finite, and let $\widehat{\Gamma}$ be a N-burst of Γ. Then the partition $\{T(i) \mid i \in I\}$ of $\widehat{I}$ is an admissible partition of $\widehat{\Gamma}$, of type (isomorphic to) Γ.*

Proof. It is enough to check that, for all $i,\ j \in I, i \neq j$, $\{T(i), T(j)\}$ is an admissible partition of $\widehat{\Gamma}_{T(i)\sqcup T(j)}$, of type $I_2(m_{i,j})$ (with $I_2(2) = A_1 \times A_1$).

If $m_{i,j} = 2$, then there is no edge between a vertex of $T(i)$ and a vertex of $T(j)$. If $m_{i,j} \in \mathbb{N}_{\geqslant 3}$, then the graph $\widehat{\Gamma}_{T(i)\sqcup T(j)}$ is the disjoint union of $\frac{2N}{m_{i,j}-1}$ copies of the spherical Coxeter graph of type $A_{m_{i,j}-1}$, and the partition $\{T(i), T(j)\}$ induces on each of these connected components the *bipartite* partition of $A_{m_{i,j}-1}$. If $m_{i,j} = \infty$, then the graph $\widehat{\Gamma}_{T(i)\sqcup T(j)}$ is the disjoint union of $\frac{N}{2}$ copies of the affine Coxeter graph of type $\tilde{A}_3$, and the partition $\{T(i), T(j)\}$ induces on each of these connected components the partition of $\tilde{A}_3$ described in Example 34 above. We conclude by applying results of [14, Sec. 2.5] recalled in Propositions 47, 49 and 50 below (note that we really need our stronger version, Proposition 49, of [14, Lem. 2.5.4] when $m_{i,j} = \infty$). $\square$

Example 38. If Γ is of type H_3 (resp. H_4), then $N_0 = 2$ and every 2-burst $\widehat{\Gamma}$ of Γ is of type D_6 (resp. E_8). We thus recover the figures of Example 17.

Remark 39. When Γ has an infinite coefficient, then $\{T(i) \mid i \in I\}$ is not an LCM-partition of $\widehat{\Gamma}$ (condition (In) of Definition 30 is not satisfied): indeed, if $m_{i,j} = \infty$, then for $i^{(k)} \in T(i)$, we get that the graph $\widehat{\Gamma}_{\{i^{(k)}\}\cup T(j)}$ is the disjoint union of $N-2$ connected components of type A_1 and one connected component of type A_3, hence is spherical.

3.4. *Some properties of admissible morphisms*

In this subsection, we show that some properties established in [3,4,9] for their special cases of admissible morphisms are in fact satisfied by all admissible morphisms.

3.4.1. *Respect of the combinatorics*

Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix, $J \subseteq I$ be a subset of I, and $\tilde{J}$ be an admissible partition of Γ_J of type $\tilde{\Gamma}$. We consider the admissible morphism $\varphi : B^+_{\tilde{\Gamma}} \hookrightarrow B^+_{\Gamma}$ induced by $\tilde{J}$, and we denote by $\tilde{\boldsymbol{W}}$ the image of the subgroup $\tilde{W} = \langle r_\alpha,\ \alpha \in \tilde{J}\rangle$ of W_Γ in $\boldsymbol{W}_\Gamma \subseteq B^+_\Gamma$.

We know that φ respects lcm's and divisibility, in the sense of Proposition 19 above. The following lemma establishes that φ respects the notions of simple elements in $B^+_{\tilde{\Gamma}}$ and in $\tilde{B}^+_\Gamma$; it is a generalization of the well-known analogous result for the standard parabolic subgroups, and of [4, Lem. 15], [3, Lem. 2.2] and [9, Prop. 2.6].

Lemma 40. *With the above notations, we get $\varphi(\boldsymbol{W}_{\tilde{\Gamma}}) = \tilde{B}^+ \cap \boldsymbol{W}_\Gamma = \tilde{\boldsymbol{W}}$. Moreover, if $\tilde{\Gamma}$ (or equivalently Γ_J) is spherical, then $\varphi(\boldsymbol{r}_{\tilde{J}}) = \boldsymbol{r}_J$.*

Proof. This is a direct consequence of Proposition 15 and Lemma 21. $\square$

Let us mention two consequences of that result, given by [9, Thm. 2.10 and Cor. 2.11], which apply to our settings; note however that for the proofs of [9, Lem. 2.9 and Thm. 2.10] to be correct, we have to add to their hypothesis the following condition, which is satisfied by any admissible morphism: $\mathrm{Im}(\varphi) \subseteq B^+_{\bigcup_{\alpha\in\tilde{J}} p(\alpha)}$, where $p(\alpha) = \{i \in I \mid \boldsymbol{s}_i \preccurlyeq \varphi(\boldsymbol{s}_\alpha)\} = \{i \in I \mid \varphi(\boldsymbol{s}_\alpha) \succcurlyeq \boldsymbol{s}_i\}$.

Propostion 41 (*[9, Thm. 2.10]*). *Let φ be as above. Then:*

(1) *the morphism φ respects (left and right) normal forms, i.e. if $(x_1, \ldots, x_n)$ is the left (resp. right) normal form of a non-trivial element $x \in B^+_{\tilde{\Gamma}}$, then $(\varphi(x_1), \ldots, \varphi(x_n))$ is the left (resp. right) normal form of $\varphi(x) \in B^+_\Gamma$,*

(2) *the morphism φ respects (left and right) gcd's, i.e. for all $(x, y) \in (B^+_{\tilde{\Gamma}})^2$, we get $\varphi(x \wedge_L y) = \varphi(x) \wedge_L \varphi(y)$ and $\varphi(x \wedge_R y) = \varphi(x) \wedge_R \varphi(y)$.*

Corollary 42 (*[9, Cor. 2.11]*). *Assume that Γ and $\tilde{\Gamma}$ are spherical. Then the morphism $\varphi_{\mathrm{gr}} : B_{\tilde{\Gamma}} \hookrightarrow B_\Gamma$ respects (left and right) irreducible fractions, i.e. if $(x, y) \in (B^+_{\tilde{\Gamma}})^2$ is the left (resp. right) irreducible form of an element $g \in B_{\tilde{\Gamma}}$, then $(\varphi(x), \varphi(y))$ is the left (resp. right) irreducible form of $\varphi_{\mathrm{gr}}(g) \in B_\Gamma$.*

3.4.2. Composition of admissible morphisms

In Proposition 43 below, we recall the result [14, Lem. 2.5.6] on *admissible partitions of an admissible partition*. This result implies that the class of admissible morphisms is closed by composition (see Corollary 44 below) and offers a criterion to test the admissibility of some spherical partitions, which we use in Example 45 below and further in Section 4.

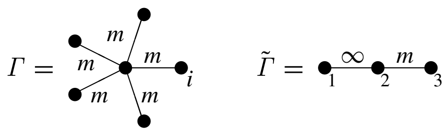
Propostion 43 (*[14, Lem. 2.5.6]*). *Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix and let I' be an admissible partition of Γ, of type Γ'. Let I'' be a spherical partition of Γ', of type Γ''. Set $\overline{\Phi} = \bigcup_{\alpha\in\Phi}\alpha$ for $\Phi \in I''$ and $\overline{I} = \{\overline{\Phi} \mid \Phi \in I''\}$. Then $\overline{I}$ is a spherical partition of Γ, of type (isomorphic to) Γ'', and $\overline{I}$ is admissible if and only if I'' is admissible.*

The following result has been established for the LCM-homomorphisms of [3] (*cf.* [3, page 134]). It can be shown that it is not true for the LCM-homomorphisms of [9].

Corollary 44. *The composition of two admissible morphisms is an admissible morphism.*

Proof. Let Γ, Γ' and Γ'' be three Coxeter matrices and let $\varphi : B^+_{\Gamma'} \to B^+_{\Gamma}$ and $\varphi' : B^+_{\Gamma''} \to B^+_{\Gamma'}$ be two admissible morphisms. In other words, Γ' is the type of an admissible partition J' of $J \subseteq I$, and Γ'' is the type of an admissible partition K'' of $K' \subseteq J'$. But K' is then an admissible partition of $K = \bigcup_{\alpha\in K'}\alpha \subseteq J$ (*cf.* Theorem 13), and Proposition 43 tells us that $\overline{K} = \{\overline{\Phi} \mid \Phi \in K''\}$ is an admissible partition of K. Moreover we get $\varphi\circ\varphi'(\boldsymbol{s}_\Phi) = \varphi(\boldsymbol{r}_\Phi) = \varphi(\mathrm{lcm}\{\boldsymbol{s}_\alpha \mid \alpha\in\Phi\}) = \mathrm{lcm}\{\varphi(\boldsymbol{s}_\alpha) \mid \alpha\in\Phi\} = \mathrm{lcm}\{\boldsymbol{r}_\alpha \mid \alpha\in\Phi\} = \boldsymbol{r}_{\overline{\Phi}}$ for every $\Phi \in K''$ (we use Proposition 19 for the third equality). Hence $\varphi\circ\varphi'$ is the admissible morphism induced by the admissible partition $\overline{K}$ of K. $\square$

Example 45. Consider the two following Coxeter graphs, where $m \in \mathbb{N}_{\geqslant 3}$:



The graph $\tilde{\Gamma}$ (which is of type FC) is the type of the admissible partition of Γ composed of orbits of Γ under the action of the automorphisms of Γ that fix the vertex i. Proposition 43 then implies that the spherical partition $\{\{1,3\},\{2\}\}$ of $\tilde{\Gamma}$ is admissible since it "lifts" to the admissible partition of Γ composed of orbits of Γ under the action of the whole group $\mathrm{Aut}(\Gamma)$. This admissible 2-partition of $\tilde{\Gamma}$ is of type $I_2(\infty)$ (since $\tilde{\Gamma}$ is not spherical) and is not an LCM-partition (condition (In) of Definition 30 is not satisfied) since $\tilde{\Gamma}_{\{2,3\}}$ is spherical.

3.4.3. Geometrical point of view

In [3, Sec. 3] (resp. in [4, Sec. 5] and in [9, Sec. 3.2]), the authors gave a geometrical interpretation of their special case of admissible morphism between Artin–Tits groups in terms of a map between the associated Salvetti complexes (resp. modified Deligne complexes). One can check that these constructions are still valid for general admissible morphisms.

However, Godelle's proof of the injectivity of LCM-homomorphisms between type FC Artin–Tits groups – more precisely the proof of [9, Prop. 3.7] – does not work for an admissible morphism between type FC Artin–Tits groups that is not an LCM-homomorphism (and such a morphism exists, *cf.* Example 45). I do not know whether such a morphism is injective or not.

4. Classification

The aim of this section is to complete the classification of admissible partitions whose type has no infinite label, began in [14, Sec. 2.5]. Thanks to our results of Section 3.3.2 above, this will in particular give us the classification of LCM-homomorphisms of [3].

The results [13, Thm. 1.2] and [14, Lem. 2.5.5] (*cf.* Theorem 13 and Proposition 15 above) reduce this classification to the classification of admissible 2-partitions of spherical Coxeter graphs. In Section 4.1, we deal with the case $|r_\alpha r_\beta| = 2$ and then recall some results of [14, Sec. 2.5] which allow us to again reduce the problem into the classification of admissible 2-partitions of irreducible spherical Coxeter graphs.

In Section 4.2, we recall the classification of admissible 2-partitions of Coxeter graphs of types A_n, B_n and D_n, obtained by Mühlherr in [14, Sec. 2.5], and complete it by examining the exceptional cases.

Finally, in Section 4.3, we compare this classification with the notion of *foldings* of a Coxeter graph, defined by Crisp in [3, Def. 4.1] in order to provide examples of LCM-homomorphisms and to begin their classification. This leads us to a generalization (and simplification) of the notion of foldings, which becomes equivalent to the notion of admissible partitions, and allows us to complete the list of cases of the original definition [3, Def. 4.1].

4.1. Admissibility and reducibility

Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix.

Using Tits' solution of the word problem [16, Thm. 3], one obtains the following result, where the *support* of $w \in W_\Gamma$ – denoted by $\mathrm{Supp}(w)$ – is the set of letters of any reduced representation of w on I (this set does not depend of the choice of the reduced representation of w since two such words only differ from a finite sequence of braid relations $\prod_{m_{i,j}}(i, j) \rightsquigarrow \prod_{m_{i,j}}(j, i)$ with $i,\ j \in I$ such that $i \neq j$ and $m_{i,j} \neq \infty$, which do not change the set of letters involved).

Lemma 46. *Let $v,\ w \in W_\Gamma$ such that $\mathrm{Supp}(v) \cap \mathrm{Supp}(w) = \emptyset$. Then:*

(1) $\ell(vw) = \ell(v) + \ell(w)$.
(2) $vw = wv \iff \forall\,(i, j) \in \mathrm{Supp}(v) \times \mathrm{Supp}(w),\ m_{i,j} = 2$,

We can now deal with the case of the admissible 2-partitions $\{\alpha, \beta\}$ of Γ with $|r_\alpha r_\beta| = 2$:

Propostion 47. *Let $\tilde{I} = \{\alpha, \beta\}$ be a spherical 2-partition of Γ. Then we have $|r_\alpha r_\beta| = 2 \Leftrightarrow \Gamma = \Gamma_\alpha \times \Gamma_\beta$. In that case, $\tilde{I}$ is an admissible partition of Γ.*

Proof. If $\Gamma = \Gamma_\alpha \times \Gamma_\beta$, then we obviously have $|r_\alpha r_\beta| = 2$. If $|r_\alpha r_\beta| = 2$, then $r_\alpha r_\beta = r_\beta r_\alpha$ and, by the previous lemma, we get that $\Gamma = \Gamma_\alpha \times \Gamma_\beta$ and $\ell(r_\alpha r_\beta) = \ell(r_\alpha) + \ell(r_\beta)$. The result [13, Lem. 3.3] (*cf.* Lemma 14 or 22 above) then implies that $\tilde{I}$ is an admissible partition of Γ. □

We will need the following proposition to limit the "forms" that an admissible 2-partition $\{\alpha, \beta\}$ of Γ can have when $|r_\alpha r_\beta| \geqslant 3$. For convenience, we sketch the proof of Mühlherr below.

Propostion 48 (*[14, Lem. 2.5.15]*). *Let $\tilde{I} = \{\alpha, \beta\}$ be an admissible 2-partition of Γ. Assume that there exists $i_0 \in \alpha$ such that $m_{i_0,j} = 2$ for all $j \in \beta$. Then $\Gamma = \Gamma_\alpha \times \Gamma_\beta$ (and hence $|r_\alpha r_\beta| = 2$).*

Proof. We have $r_\beta s_{i_0} = s_{i_0} r_\beta$, so we get by Lemma 46 (first assertion) that $\ell(r_\alpha r_\beta s_{i_0}) = \ell(r_\alpha s_{i_0} r_\beta) = \ell(r_\alpha s_{i_0}) + \ell(r_\beta) = \ell(r_\alpha) - 1 + \ell(r_\beta) = \ell(r_\alpha r_\beta) - 1$, *i.e.* $i_0 \in I^-(r_\alpha r_\beta)$. Since $\tilde{I}$ is admissible, we then have $\alpha \subseteq I^-(r_\alpha r_\beta)$, whence $I = \alpha \cup \beta \subseteq I^-(r_\alpha r_\beta)$ and $r_\alpha r_\beta = r_I = r_\beta r_\alpha$. We conclude by Lemma 46 (second assertion). □

The following proposition allows us to reduce our classification problem to the irreducible case. It is given in [14, Lem. 2.5.4] for *spherical* Coxeter graphs $\Gamma_1, \ldots, \Gamma_n$, but in order to complete the proof of Theorem 37 above, we need it for general Coxeter graphs. So we prove it below in this more general context, using our characterizations of the admissibility of a 2-partition of Γ in terms of simple elements in B^+_Γ (*cf.* Lemma 22).

Propostion 49 (*[14, Lem. 2.5.4]*). *Assume that $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$. For $1 \leqslant k \leqslant n$, let $\{\alpha_k, \beta_k\}$ be a spherical 2-partition of Γ_k and set $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_n$ and $\beta = \beta_1 \sqcup \cdots \sqcup \beta_n$. Then $\{\alpha, \beta\}$ is a spherical 2-partition of Γ with $r_\alpha = r_{\alpha_1} \cdots r_{\alpha_n}$, $r_\beta = r_{\beta_1} \cdots r_{\beta_n}$ and $|r_\alpha r_\beta| = \mathrm{lcm}\{|r_{\alpha_k} r_{\beta_k}| \mid 1 \leqslant k \leqslant n\}$. Moreover, the following conditions are equivalent:*

(1) *$\{\alpha, \beta\}$ is an admissible partition of Γ,*
(2) *$\{\alpha_k, \beta_k\}$ is an admissible partition of Γ_k for $1 \leqslant k \leqslant n$, and $|r_{\alpha_1} r_{\beta_1}| = |r_{\alpha_2} r_{\beta_2}| = \cdots = |r_{\alpha_n} r_{\beta_n}|$.*

In that case, we get $|r_\alpha r_\beta| = |r_{\alpha_1} r_{\beta_1}| = |r_{\alpha_2} r_{\beta_2}| = \cdots = |r_{\alpha_n} r_{\beta_n}|$.

Proof. It is enough to prove the result for $n = 2$. The first observations are clear (if needed with the help of Lemma 46). Note that, thanks to Lemma 46, we get that $\boldsymbol{W}_\Gamma \approx \boldsymbol{W}_{\Gamma_1} \times \boldsymbol{W}_{\Gamma_2}$. For example, we have $\boldsymbol{r}_\alpha = \boldsymbol{r}_{\alpha_1}\boldsymbol{r}_{\alpha_2}$ and $\boldsymbol{r}_\beta = \boldsymbol{r}_{\beta_1}\boldsymbol{r}_{\beta_2}$. Hence, for all $m \in \mathbb{N}$, we get $\prod_m(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta) = \prod_m(\boldsymbol{r}_{\alpha_1}, \boldsymbol{r}_{\beta_1})\prod_m(\boldsymbol{r}_{\alpha_2}, \boldsymbol{r}_{\beta_2})$ in $B^+_\Gamma \approx B^+_{\Gamma_1} \times B^+_{\Gamma_2}$.

Suppose (2) and let us show (1). We get $|r_\alpha r_\beta| = |r_{\alpha_1} r_{\beta_1}| = |r_{\alpha_2} r_{\beta_2}|$. For $k = 1, 2$, Lemma 22 gives us that $\prod_m(\boldsymbol{r}_{\alpha_k}, \boldsymbol{r}_{\beta_k})$ and $\prod_m(\boldsymbol{r}_{\beta_k}, \boldsymbol{r}_{\alpha_k})$ are simple for all $0 \leqslant m < |r_{\alpha_k} r_{\beta_k}| + 1$. Then $\prod_m(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta)$ and $\prod_m(\boldsymbol{r}_\beta, \boldsymbol{r}_\alpha)$ are simple for all $0 \leqslant m < |r_\alpha r_\beta| + 1$ and we are done by applying Lemma 22 again.

Suppose (1) and let us show (2). If $|r_\alpha r_\beta| \neq \infty$, then necessarily $|r_{\alpha_k} r_{\beta_k}| \neq \infty$ for $k = 1, 2$. Moreover, Γ is then spherical (by Proposition 15), hence so is Γ_k for $k = 1, 2$. Lemma 22 gives us that $\boldsymbol{r}_I = \prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta) = \prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_\beta, \boldsymbol{r}_\alpha)$. Let us denote by I_k the vertex set of Γ_k for $k = 1, 2$. Since we have $\boldsymbol{r}_I = \boldsymbol{r}_{I_1}\boldsymbol{r}_{I_2}$, $\prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta) = \prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_{\alpha_1}, \boldsymbol{r}_{\beta_1})\prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_{\alpha_2}, \boldsymbol{r}_{\beta_2})$, and similarly if we exchange the roles of α and β, and the roles of a_k and β_k, we conclude, by identifying the terms in $B^+_{\Gamma_1}$ and $B^+_{\Gamma_2}$, that $\boldsymbol{r}_{I_k} = \prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_{\alpha_k}, \boldsymbol{r}_{\beta_k}) = \prod_{|r_\alpha r_\beta|}(\boldsymbol{r}_{\beta_k}, \boldsymbol{r}_{\alpha_k})$, for $k = 1, 2$. If $|r_\alpha r_\beta| = \infty$, then Lemma 22 shows us that the element $\prod_m(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta) = \prod_m(\boldsymbol{r}_{\alpha_1}, \boldsymbol{r}_{\beta_1})\prod_m(\boldsymbol{r}_{\alpha_2}, \boldsymbol{r}_{\beta_2})$ is simple for all $m \in \mathbb{N}$, and similarly if we exchange the roles of α and β, and the roles of a_k and β_k. We then have that $\prod_m(\boldsymbol{r}_{\alpha_k}, \boldsymbol{r}_{\beta_k})$ and $\prod_m(\boldsymbol{r}_{\beta_k}, \boldsymbol{r}_{\alpha_k})$ are simple for all $m \in \mathbb{N}$ (and $k = 1, 2$). In both cases, Lemma 32 shows that $|r_\alpha r_\beta| = |r_{\alpha_k} r_{\beta_k}|$, for $k = 1, 2$, and we conclude thanks to Lemma 22. $\square$

4.2. *Admissible 2-partitions of irreducible spherical Coxeter graphs*

Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a spherical Coxeter matrix and let $\tilde{I} = \{\alpha, \beta\}$ be a 2-partition of Γ. Let us denote by $\Gamma_1, \ldots, \Gamma_n$ the connected components of Γ, and by I_k the vertex set of Γ_k for $1 \leqslant k \leqslant n$.

- If there exists $1 \leqslant k \leqslant n$ such that I_k is included in α or in β, then $\tilde{I}$ is admissible if and only if $\Gamma = \Gamma_\alpha \times \Gamma_\beta$, in which case $|r_\alpha r_\beta| = 2$ (by Proposition 48).
- If not, then ($|r_\alpha r_\beta| \neq 2$ and) α and β meet every connected component of Γ, and we are in the situation of Proposition 49, with $\alpha_k = \alpha \cap I_k$ and $\beta_k = \beta \cap I_k$ for $1 \leqslant k \leqslant n$. So we get that $\tilde{I} = \{\alpha, \beta\}$ is admissible if and only if $\{\alpha_k, \beta_k\}$ is an admissible 2-partition of Γ_k for $1 \leqslant k \leqslant n$, and $|r_\alpha r_\beta| = |r_{\alpha_1} r_{\beta_1}| = |r_{\alpha_2} r_{\beta_2}| = \cdots = |r_{\alpha_n} r_{\beta_n}|$.

Hence we are left with the classification of admissible 2-partitions of irreducible spherical Coxeter graphs and their corresponding coefficient $|r_\alpha r_\beta|$. The first result in this direction in the following proposition. Since irreducible spherical Coxeter graphs are finite trees (hence bipartite), each of them has a unique *bipartite* partition, which is a 2-partition except for the type A_1.

Propostion 50 (*[14, Lem. 2.5.13]*). *The bipartite partition $\{\alpha, \beta\}$ of an irreducible spherical Coxeter graph (distinct from A_1) is admissible, and the coefficient $|r_\alpha r_\beta|$ is the Coxeter number of the graph.*

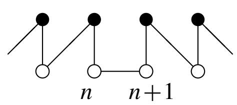
Proof. The result [2, Lem. 5.8] gives our characterization of Lemma 22. $\square$

Remark 51. These considerations justify all cases of Examples 16 and 17 above except the ones concerning the non-trivial automorphisms of A_{2n}, F_4 and $I_2(m)$ (this last one being obvious), and reduce the justifications for A_{2n} to the A_4 case. These last two cases (non-trivial automorphisms of A_4 and F_4) can be dealt with by direct computations.

Let us now investigate the different situations case-by-case.

4.2.1. *Admissible 2-partitions of A_n, B_n, D_n*

The admissible 2-partitions of Coxeter graphs of type A_n ($n \geqslant 2$), B_n ($n \geqslant 2$) and D_n ($n \geqslant 4$) have been classified by Mühlherr in [14, Sec. 2.5]. In those cases, the only admissible 2-partitions are the bipartite ones and, for every $n \geqslant 2$, the following 2-partition of A_{2n} (where the vertices are numbered in the natural order):

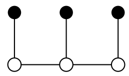
(1) $\cdots$  $\cdots$ with $|r_\alpha r_\beta| = 2n$.

The admissibility of this 2-partition is a consequence of [14, Lem. 2.5.6] (*cf.* Proposition 43 above) applied to the admissible partition of A_{2n} induced by its non-trivial automorphism and the bipartite partition of B_n.

Mühlherr first established the classification for the A_n case by explicit computations in the symmetric group. He inferred from this the classification for the B_n case using [14, Lem. 2.5.6], which shows that every admissible 2-partition of B_n "lifts" to an admissible 2-partition of A_{2n} (or A_{2n-1}). In the same vein, since the automorphism of D_n that permutes the vertices $n-1$ and n (for the standard numbering of [1, Planche IV]) gives an admissible partition of type B_{n-1}, and since [14, Lem. 2.5.15] (*cf.* Proposition 48 above) shows that for every admissible 2-partition of D_n, the vertices $n-1$ and n must be in the same part of the partition, we get by [14, Lem. 2.5.6] that every admissible 2-partition of D_n induces an admissible 2-partition of B_{n-1}, whence the classification for the D_n case.

4.2.2. Admissible 2-partitions of E_6, E_7 and E_8

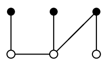
Mühlherr showed in [14, Lem. 2.5.14] that the following 2-partition of E_6 is admissible: this is a consequence of [14, Lem. 2.5.6] (*cf.* Proposition 43 above) applied to the admissible partitions of E_6 and F_4 induced by their non-trivial automorphism.

(2)  with $|r_\alpha r_\beta| = 8$

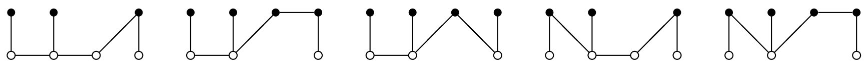
Propostion 52. *The only admissible 2-partitions of the Coxeter graphs E_n $(n = 6, 7, 8)$ are the bipartite ones and the 2-partition* (2) *above.*

Proof. Let Γ be a Coxeter graph of type E_6, E_7 or E_8 and let $\{\alpha, \beta\}$ be an admissible 2-partition of Γ. Since Γ is connected, $\{\alpha, \beta\}$ does not satisfy the condition of Proposition 48 above. Hence, apart from the bipartite partitions and the 2-partition (2) above, there are fifteen other possibilities:

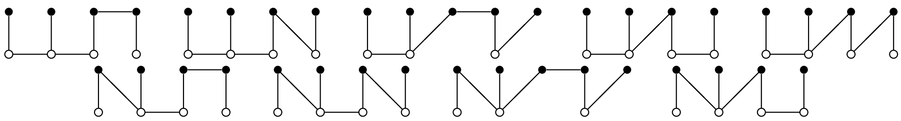
- one for E_6:



- five for E_7:



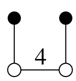
- and nine for E_8:



By Lemma 14, there exist $n \in \mathbb{N}$ such that $\prod_n(r_\alpha, r_\beta) = \prod_n(r_\beta, r_\alpha) = r_I$ and $\sum_n(\ell(r_\alpha), \ell(r_\beta)) = \sum_n(\ell(r_\beta), \ell(r_\alpha)) = \ell(r_I)$. Since we have $\ell(r_I) = 36$ (resp. 63, resp. 120) if $\Gamma = E_6$ (resp. E_7, resp. E_8), *cf.* [1, Planches V-VII], the consideration on lengths eliminates the last candidate for E_6 and leaves only one candidate for E_7 (the second one, with $n = 14$) and four for E_8 (the first one with $n = 20$, and the third, fourth and sixth ones with $n = 24$). We then verify, if needed with the help of a computation software like GAP or Maple, that the equality $\prod_n(r_\alpha, r_\beta) = \prod_n(r_\beta, r_\alpha) = r_I$ occurs in none of the five remaining cases, hence those 2-partitions are not admissible. □

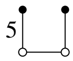
4.2.3. Admissible 2-partitions of F_4, H_3, H_4 (and $I_2(m)$, $m \geqslant 3$)

The orbits of F_4 under the action of its non-trivial automorphism form the following admissible 2-partition:

(3)  with $|r_\alpha r_\beta| = 8$

Propostion 53. *The only admissible 2-partitions of the Coxeter graphs F_4, H_3, H_4 and $I_2(m)$, $m \geqslant 3$, are the bipartite ones and the 2-partition* (3) *above.*

Proof. There is nothing to prove for the dihedral graphs. So assume that Γ is a Coxeter graph of type F_4, H_3 or H_4, and let $\{\alpha, \beta\}$ be an admissible 2-partition of Γ. Since Γ is connected, $\{\alpha, \beta\}$ does not satisfy the condition of Proposition 48 above and hence is either a bipartite partition, or the 2-partition (3) above, or possibly the following 2-partition of H_4:



To show that this last 2-partition is non-admissible, one can follow the same lines as in the proof of Proposition 52. Otherwise, note that H_4 is the type of an admissible partition of E_8 (*cf.* Example 17 or 38) so, thanks to Proposition 43, the admissibility of the above 2-partition of H_4 is equivalent to the admissibility of a certain 2-partition of E_8 (not the bipartite one), which has been shown to be non-admissible in Proposition 52. $\square$

4.3. Foldings

Let $\Gamma = (m_{i,j})_{i,j\in I}$ and $\Gamma' = (m'_{i',j'})_{i',j'\in I'}$ be two Coxeter matrices with no infinite coefficient. Crisp defined in [3, Def. 4.1] the notion of a *folding* of Γ' onto Γ, in order to give examples of LCM-homomorphisms and to begin their classification. With our terminology, a folding of Γ' onto Γ is a surjective map $f : I' \twoheadrightarrow I$ that satisfy a list of conditions made for the partition $\{f^{-1}(\{i\}) \mid i \in I\}$ of I' to be an LCM-partition of type (isomorphic to) Γ [3, Prop. 4.2]. Crisp concluded [3, Sec. 4] by asking essentially if every LCM-partition is obtained from a folding. The classification we have just established shows that the answer is no, with the definition [3, Def. 4.1] for a folding, and indicates how to complete the list of cases of [3, Def. 4.1] to turn the answer to yes.

In Definition 54 below, we propose a generalization of the notion of foldings that fit to our new point of view, and in Proposition 56, we rephrase in the manner of [3, Def. 4.1] the classification established above.

Definition 54 (*Foldings*). Let $\Gamma = (m_{i,j})_{i,j\in I}$ and $\Gamma' = (m'_{i',j'})_{i',j'\in I'}$ be two Coxeter matrices. A *folding* of Γ' onto Γ is a map $f : I' \to I$ such that the set $\{f^{-1}(\{i\}) \mid i \in I\}$ is an admissible partition of Γ', of type (isomorphic to) Γ.

Notation 55. Let Γ be any Coxeter graph. For $n \in \mathbb{N}_{\geqslant 1}$, we denote by $n\Gamma$ the disjoint union of n copies of Γ.

Propostion 56. *Let $\Gamma = (m_{i,j})_{i,j\in I}$ and $\Gamma' = (m'_{i',j'})_{i',j'\in I'}$ be two Coxeter matrices and $f : I' \to I$ be any map from I' to I. Assume that Γ has no infinite coefficient. Then f is a folding from Γ' onto Γ if and only if f satisfies the following conditions for every $i, j \in I$:*

(1) *the subset $f^{-1}(\{i\})$ of I' is non-empty and spherical,*

(2) *if $m_{i,j} = 2$, then there is no edge between a vertex of $f^{-1}(\{i\})$ and a vertex of $f^{-1}(\{j\})$, i.e. $\Gamma'_{f^{-1}(\{i,j\})} = \Gamma'_{f^{-1}(\{i\})} \times \Gamma'_{f^{-1}(\{j\})}$,*

(3) *if $m_{i,j} \geqslant 3$, then one of the following occurs:*

- **(A)** *$\Gamma'_{f^{-1}(\{i,j\})} = nI_2(m_{i,j})$ for some $n \in \mathbb{N}_{\geqslant 1}$, and each connected component of $\Gamma'_{f^{-1}(\{i,j\})}$ (of type $I_2(m_{i,j})$) meets $f^{-1}(\{i\})$ and $f^{-1}(\{j\})$,*
- **(B)** *$\Gamma'_{f^{-1}(\{i,j\})}$ is an irreducible and spherical Coxeter graph with Coxeter number $m_{i,j}$, and the 2-partition $\{f^{-1}(\{i\}), f^{-1}(\{j\})\}$ of $f^{-1}(\{i,j\})$ is the bipartite partition of $\Gamma'_{f^{-1}(\{i,j\})}$,*
- **(C1)** *$m_{i,j} = 2n$ for some $n \in \mathbb{N}_{\geqslant 2}$, $\Gamma'_{f^{-1}(\{i,j\})} = A_{2n}$, and the 2-partition $\{f^{-1}(\{i\}), f^{-1}(\{j\})\}$ of $f^{-1}(\{i,j\})$ is the admissible 2-partition* (1) *of Section 4.2.1,*
- **(C2)** *$m_{i,j} = 8$, $\Gamma'_{f^{-1}(\{i,j\})} = E_6$, and the 2-partition $\{f^{-1}(\{i\}), f^{-1}(\{j\})\}$ of $f^{-1}(\{i,j\})$ is the admissible 2-partition* (2) *of Section 4.2.2,*
- **(C3)** *$m_{i,j} = 8$, $\Gamma'_{f^{-1}(\{i,j\})} = F_4$, and the 2-partition $\{f^{-1}(\{i\}), f^{-1}(\{j\})\}$ of $f^{-1}(\{i,j\})$ is the admissible 2-partition* (3) *of Section 4.2.3,*
- **(D)** *the map $f^{-1}(\{i,j\}) \to \{i,j\}$ induced by f is a composition $h \circ g$, where g is a folding from $\Gamma'_{f^{-1}(\{i,j\})}$ onto $nI_2(m_{i,j})$ ($n \in \mathbb{N}_{\geqslant 2}$) defined only with cases* (B) *to* (C3) *and h is a folding from $nI_2(m_{i,j})$ onto $\Gamma_{\{i,j\}} = I_2(m_{i,j})$ of case* (A).

Proof. This is a reformulation of the classification obtained above. $\square$

Remark 57. We have added to the list of [3, Def. 4.1] the cases (C1) for $n > 2$, (C2) and (C3). Note that [10, Def. 1.11] already includes case (C3).

Remark 58. The cases (A) to (D) imply, for a non-isolated vertex i of Γ, that $\Gamma'_{f^{-1}(\{i\})}$ is non-empty and spherical, hence our condition (1) can be relaxed to the weaker condition (implicit in [3, Def. 4.1] and [10, Def. 1.11]):

(1$'$) if i is isolated in Γ, then $f^{-1}(\{i\})$ is non-empty and spherical.

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